Structural controllability of highway networks

Shima Sadat Mousavi

Kimia Chavoshi

Anastasios Kouvelas

Conference paper STRC 2020

STRC | 20th Swiss Transport Research Conference
Monte Verità / Ascona, May 13 – 15, 2020

Structural controllability of highway networks

Shima Sadat Mousavi
IVT
ETH Zürich
CH-8093 Zurich
shimaossadat.mousavi@ivt.baug.ethz.ch

Kimia Chavoshi
IVT
ETH Zurich
kimia.chavosi@ivt.baug.ethz.ch

Anastasios Kouvelas
IVT
ETH Zurich
anastasios.kouvelas@ivt.baug.ethz.ch

Abstract

The previous studies have introduced useful methods to control the traffic flow on highways via actuators such as Ramp Metering (RM) and Variable Speed Limits (VSL). In this direction, in order to minimize the installation costs, determining the required number of actuators and their locations is crucial for the network control. With this in mind, In this paper, by considering a linear dynamics for the highway networks, we utilize structural controllability techniques to identify a minimal set of controllers which renders the network controllable. This approach provides a notion of robustness with respect to uncertainties in different parameters of the system, which can ensure the controllability of the network even in the lack of an exact dynamics of the highway network.

Keywords

Highway Management, Structural Controllability, Nonlinear Model Predictive Control

1 Introduction

Nowadays, we encounter with increasing demand for transport on metropolitan areas that consequently increases the congestion level on highway networks. During the past decades, a surge of interest in freeway traffic has arisen due to its crucial impact on the safety, the economy, and the environment. There are evidences that one can ameliorate the traffic congestion by designing and employing an appropriate traffic flow management. Prior studies have proposed different approaches to improve traffic conditions on highway networks. Ramp Meterings (RM's) and Variable Speed Limits (VSL's) have been widely considered as two effective actuators to regulate the traffic flow on the highways. The RM methods provide better traffic conditions through controlling the on-ramp inflow on the freeway mainstream. Meanwhile, the VSL is another actuator that can provide more direct and efficient control of traffic flow on the mainline of a freeway compared to an RM.

In recent decades, a huge number of works in the literature have focused on studying the controllability of a general network Mousavi *et al.* (2018, 2021a,b). The network controllability problem can be defined as finding the minimum number and the locations of the actuators that can render the whole system controllable. A system is called controllable if an appropriate input can be designed such that the states of the system can be steered from any initial states toward any desired ones within a finite time. The controllability problem of a traffic network can be established to identify the location for the minimum number of ramp meters such that the whole stretch would be controllable Mousavi and Kouvelas (2020); Agarwal *et al.* (2015); Bekiaris-Liberis *et al.* (2017b,a); Rinaldi (2018).

As a dual problem of the controllability analysis of transportation networks, the observability problem that can be interpreted as identifying the location of sensors has been studied in a few work as Castillo *et al.* (2008, 2010, 2014); Agarwal *et al.* (2015); Contreras *et al.* (2015); Bekiaris-Liberis *et al.* (2017b,a).

n most of these works, some algebraic rank matrix conditions have been utilized to analyze the observability of a traffic model.

For example, in Contreras *et al.* (2015), a lumped parameter-based model for traffic dynamics on a highway segment is considered. The model is then linearized about the steady state flows, and then by calculating the observability matrix, the observability of the system for different locations of sensors is evaluated. We also note that in Contreras *et al.* (2015), the fundamental diagrams of the highway segments are assumed to be available, and the value of system parameters that appear in the set of differential equations are known. This assumption is not necessarily true in practice, since, in many cases, the exact value of the system parameters is unknown or highly uncertain; moreover, the system parameters may not be fixed and may vary over time. In addition, for large-scale networks with a large number of states, checking the algebraic condition through computing the rank of very large matrices is not numerically feasible.

The above-mentioned reasons motivate us to study the structural controllability of traffic networks from a merely graphical point of view. In this framework, only the zero-nonzero pattern of system matrices which describes the wiring diagram or the topology of the network is of interest. We also note that, in Contreras *et al.* (2015), to simplify the computation of the equilibrium points, all the highway segments are assumed to have the same traffic features, such as fundamental diagrams, free flow speeds, and jam densities. With such a consideration, the heterogeneity of different segments of a highway or some traffic events like lane-drop cannot be properly modelled; while in this work, a general model without any restrictive assumption on the parameter values is studied.

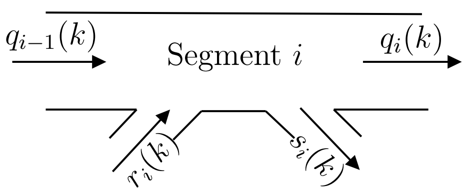
Various methods have been proposed in literature to manipulate RM and VSL. RM strategies can be categorized as local and coordinated. Local RM regulates the on-ramp inflow based on its neighbourhood traffic information, in order to ameliorate the local traffic conditions. ALINEA, presented in Papageorgiou *et al.* (1991), is one of the most popular methods in this category, that employs a feedback control method. Although local ramp metering is well-known and widely used, especially due to its simplicity, it is easy to show that when looking at a bigger area it is sub-optimal and can be outperformed by coordinated RM strategies. In practice, local strategies could demonstrate poor performance, and one of the main reasons is the presence of limited ramp storage space.On the other hand, Coordinated methods regulate the on-ramps inflows by utilizing system-wide traffic information, in order to enhance the overall network performance. Various methods have been presented in the literature for coordinated ramp metering; for instance Lu *et al.* (2017) applied a reinforcement learning method to deal with equity issues (i.e. users from different on-ramps have equal access to the mainline). Moreover, Haj-Salem *et al.* (2018) presented a multi-objective nonlinear optimization that includes two cost functions, for traffic and safety (based on a risk index model). A model predictive hierarchical control method is developed in Papamichail *et al.* (2010), where the structure is composed from an estimation, an optimization, and a direct control layers, with focus on optimizing the total time spent.
Note that various RM methods have been frequently implemented on motorways around the world. Nevertheless, the efficiency of these strategies deteriorates when they have to

deal with high demands, or in cases of considering the equitable allocation of benefits among users (see Kotsialos *et al.* (2001) for more details). Many different methods have been studied that apply VSL in order to improve freeway traffic conditions. For instance, Hegyi *et al.* (2005b) has proposed a model predictive control (MPC) approach. An optimal control strategy based on minimization of L^2 quadratic error to the desired outflow was developed by Monache *et al.* (2017). In another work, Khondaker and Kattan (2015) studied VSL for a case study with connected vehicles, and designed a multi-objective optimization function to simultaneously optimize mobility, safety and environmental sustainability.

The integration of RM and VSL methods provides the opportunity to control mainline traffic flow conditions, while, at the same time, considering the equity and on-ramp queuing aspects. The study in Zhang and Ioannou (2017) developed a feedback linearization VSL approach, coordinated with ALINEA/Q as a RM strategy, in order to maximize the flow rate and manage the on-ramp queues simultaneously. The combination of ALINEA and HERO as local and coordinated RM, respectively, together with a VSL algorithm is proposed by Li *et al.* (2014). Furthermore, the work in Hegyi *et al.* (2005a) developed an MPC approach to compute the optimal coordination of RM and VSL. Although the proposed control methods for the combination of RM and VSL have presented promising results, this problem still requires more attention. The structure of the model that is considered in this MPC approach leads to a recursive Mixed Integer Non-Linear Problem (MINLP) optimization. By increasing the number of variables in MINLP, the computational time increases with an exponential rate. Consequently, the benchmark network that has been studied in the aforementioned paper is a simplified network that consists only of one on-ramp. Chavoshi and Kouvelas (2020) modified the model by using an approximation method that simplifies the optimization from MINLP to a Non-Linear Problem (NLP). This simplification accelerates the computational time and make the NLMPC approach more practical for real world application. Furthermore, in Chavoshi and Kouvelas (2019) a feedback linearization approach has been proposed that can tackle the problem efficiently and provide computationally feasible solutions.

The rest of the paper is organized as follow: In section 2, we elaborate on the macroscopic traffic model that we utilized in this project. Section 3 is dedicated to controllability analysis. In this section, we first review the controllability notion. Afterwards, we study the structural controllability for the ring road highway networks. Through the results of controllability analysis we can find the minimum essential number of actuators (VSL and RM) and their locations in the network. In this study, the traffic management objective is to tackle the congestion issue by designing a control method for the coordination of RM and VSL. The developed coordinated control is presented in section 4. For the evaluation of the proposed methodology, we simulate the traffic flow in a large network. Antwerp ring

Figure 1: The general structure of freeway segment i.



road is selected as the case study. Section 5 contains the information of the simulation setup. Finally, the future work is presented in section 6.

2 Macroscopic traffic model

Many models have been developed to describe traffic flow in the macroscopic level. They model the traffic flow by formulating the dynamics of fundamental macroscopic traffic characteristics such as flow, density, and speed. According to the level of partial derivatives that the models cover, they are categorized as first order, second order and higher order models. METANET is one of the macroscopic second order traffic flow models, which is widely used for freeway networks. It represents the dynamics of traffic flow in terms of density and speed in the discretized time and space domains.Figure 1 shows Supposing a general segment i (discretized space that includes one on-ramp and one off-ramp) of a freeway with length Δ_i and λ_i number of lanes at a discrete time step k. We denote with $q_{i-1}(k)$ the inflow that enters segment i from the upstream segment, and similarly, $q_i(k)$ denotes the outflow of segment i that moves to the downstream segment. The variables $r_i(k)$ and $s_i(k)$ denote the on-ramp outflow and the off-ramp inflow, respectively. METANET express the dynamics of the traffic flow in segment i as follows Carlson *et al.* (2010):

$$\rho_i(k+1) = \rho_i(k) + \frac{T}{\Delta_i \lambda_i}\Big(q_{i-1}(k) - q_i(k) + r_i(k) - s_i(k)\Big), \tag{1}$$

$$s_i(k) = \beta_i(k) q_i(k), \tag{2}$$

$$v_i(k+1) = v_i(k) + \frac{T}{\tau}(V(\rho_i(k)) - v_i(k)) + \frac{T}{\Delta_i} v_i(k)(v_{i-1}(k) - v_i(k)) - \frac{vT}{\tau\Delta_i}\frac{\rho_{i+1}(k) - \rho_i(k)}{\rho_i(k) + K}$$
$$- \frac{\delta T}{\Delta_i \lambda_i}\frac{r_i(k) v_i(k)}{\rho_i(k) + K}, \tag{3}$$

$$V(\rho_i(k)) = v_{f,i} \exp\left(\frac{-1}{a_i}\left(\frac{\rho_i(k)}{\rho_{cr,i}}\right)^{a_i}\right), \tag{4}$$

$$q_i(k) = \rho_i(k) v_i(k) \lambda_i, \tag{5}$$

where $\rho_i(k)$ (veh/km/lane), $q_i(k)$ (veh/h) and $v_i(k)$ (km/h) denote the traffic density, flow, and speed, respectively, in segment i at time step k. Based on (1), also known as conservation equation, the difference between total input flows (i.e. $q_{i-1}(k)$ and $r_i(k)$) and total amount of output flows (i.e. $q_i(k)$ and $s_i(k)$) in segment i, results in the change of density $\rho_i(k)$. The exiting rate $\beta_i(k)$ denotes the ratio of $s_i(k)$ to $q_{i-1}(k)$. Equation (3) corresponds to the dynamics of speed and is composed by four different terms. The first is the so-called relaxation term, which demonstrates the tendency of vehicles to achieve the desired speed (i.e. the stationary speed $V(\rho_i(k))$). The second and third terms model the impact of spatial heterogeneity. More precisely, the second term is the so-called convection term, which expresses the effect of inflow, and the third one the so-called anticipation term that models the effect of upcoming change in density. The fourth term represents the speed drop caused by merging phenomena in case of including an on-ramp. The variable T is the time step (in sec) and τ, v, K, and δ denote model parameters that tune the weight of these four parts in the speed dynamics. The relationship between stationary speed and traffic flow is described by the well-known Fundamental Diagram (FD), demonstrated in equation (18), where $v_{f,i}$ and $\rho_{cr,i}$ represent the free flow speed and critical density, respectively. For a road segment i that does not include an on-ramp, and for the case of stationary and spatial homogeneity, the maximum observable flow $q_{\max,i}$ (which occurs for $\rho_{cr,i}$) can be derived from the above equations as follows:

$$q_{\max,i} = \rho_{cr,i} v_{f,i} \exp\left(\frac{-1}{a_i}\right). \tag{6}$$

The original METANET model does not represent the effects of applying VSL and RM on freeway; therefore, we modify the METANET to describe impacts of VSL and RM utilization.

2.1 VSL modification on METANET model

There are different models that express the effects of utilizing VSL on freeways traffic condition. For instance, Papamichail *et al.* (2010) models this impact on the FD through introducing an augmented version of equation (18), where $v_{f,i}$, $\rho_{cr,i}$, and a_i are functions of the VSL rate. This augmented model succeeds in reporting the impacts of VSL in case of active VSL functioning in the highway network. Nevertheless, its accuracy deteriorates when facing situations that the actual speed is less than the speed limit, since this model applies drastic changes to the FD even for the congested conditions, where VSL does not impose an active boundary. In order to overcome this problem and make a more realistic representation of FD, Hegyi et al. in Hegyi *et al.* (2005a) proposed an extended version of equation (18) as follows:

$$V(\rho_i(k)) = \min\left((1+\alpha)v_{\mathrm{VSL},i}(k), v_{f,i}\exp\left(\frac{-1}{a_i}\left(\frac{\rho_i(k)}{\rho_{cr,i}}\right)^{a_i}\right)\right) \tag{7}$$

where $v_{\mathrm{VSL},i}(k)$ is the speed limit on segment i. The term $(1+\alpha)$ denotes the non-compliance factor that models the disobedience of drivers towards speed limits. According to the above equation, the desired speed is the minimum between the limited speed caused by VSL and the desired speed derived from the FD without considering the VSL impact.

2.2 RM modification on METANET model

The inflow of the non-equipped on-ramps is derived by considering three terms. The first term is Q_0 and denotes the on-ramp flow capacity due to the physical characteristics of the infrastructure (i.e. number of lanes). The second term, also called supply of space, represents the effect of the mainline congestion. Finally, the third term, also known as demand for space, represents the actual demand flow and is composed by the new arrivals $d_i(k)$ (veh/h) and the vehicles already waiting in the ramp queue $w_i(k)$ (veh). According to equation (8), for the RM equipped on-ramps, the inflow to the mainline is a portion of outflow in absence of RM. That is formulated as:

$$r_i(k) = c_i(k) \times \min\Big(Q_0, Q_0\frac{\rho_{\max,i}-\rho_i(k)}{\rho_{\max,i}-\rho_{cr,i}}, d_i(k)+\frac{w_i(k)}{T}\Big), \tag{8}$$

$$w_i(k+1) = w_i(k) + T(d_i(k) - r_i(k)), \tag{9}$$

where $c_i(k)$ is the metering rate that is bounded by $c_i(k) \in [c_{\min}, 1]$, with $c_{\min}$ denoting the minimum admissible value. Finally, the dynamics of the on-ramp queue length $w_i(k)$ are presented in equation (9), where $d_i(k)$ denotes the on-ramp demand flow (veh/h).

2.3 Dynamical System Definition

We summarize the METANET model described above with the form of a dynamical system as follows:

$$X(k+1) = f(X(k), U(k), D(k)) \tag{10}$$

$$Y(k) = h(X(k)) \tag{11}$$

$$\forall i, 1 \leq i \leq N \tag{12}$$

$$X(k) = [\rho_i(k), v_i(k), w_i(k)]^T \tag{13}$$

$$U(k) = [v_{VSL,i}(k), c_i(k)]^T \tag{14}$$

$$D(k) = [d_i(k), \beta_i(k)]^T \tag{15}$$

$$Y(k) = [\rho_i(k), v_i(k)]^T \tag{16}$$

where, N is the number of segments; $X(k)$ is the system's state vector which consists of the densities, mean speeds, and queue lengths of all freeway segments; $U(k)$ is the vector with the input (control) signals, which contains speed limits and metering rates of the whole network. Note that the on-ramp demands and off-ramp exiting rates are considered disturbances and are denoted with the vector $D(k)$. Finally, $Y(k)$ is the output vector that is assumed the same as the state vector. Based on the presented METANET model $f(\cdot)$ is a nonlinear vector function with a high order of complexity.

3 Controllability analysis

The controllability analysis help us determining the minimum required number of actuators and their exact locations to dominate traffic flow on highway networks. In particular, controllability analysis is more beneficial for large networks. Since the large number of optimization's variables and thereby, the high computational cost has been always a barrier to apply on-line optimization methods for large networks. Employing the results of controllability analysis simplifies the optimization problem and enables us to deal with large networks in real time.

Although METANET is an accurate macroscopic traffic flow model, it is a nonlinear model with the high level of complexity that consequently requires cumbersome analysis for controllability and observability studies.

In order to facilitate these studies, we consider a simplified model for traffic flow. The new model which is defined based on the Lighthill-Whitman-Richards (LWR) model, describes the density's dynamics with the conservation law as follows:

$$\rho_i(k+1) = \rho_i(k) + \frac{T}{\Delta_i \lambda_i}\Big(q_{i-1}(k) - q_i(k) + r_i(k) - s_i(k)\Big), \tag{17}$$

where the the relationship between macroscopic variables, i.e., density ρ_i, flow q_i, and speed v_i can be represented by fundamental diagrams. In the simplified model, we ignore the speed dynamics, and we only define the average traffic speed as

$$v_i(\rho_i(k)) = v_{f,i} \exp\left(\frac{-1}{a_i}\left(\frac{\rho_i(k)}{\rho_{cr,i}}\right)^{a_i}\right), \tag{18}$$

Then, based on the equation $q_i(k) = \rho_i(k)v_i(k)\lambda_i$, the flow q_i can be written as a nonlinear function of the density as:

$$q_i(k) = h(\rho_i(k)) = v_{f,i}\rho_i(k)\lambda_i \exp\left(\frac{-1}{a_i}\left(\frac{\rho_i(k)}{\rho_{cr,i}}\right)^{a_i}\right). \tag{19}$$

Moreover, the dynamics of the density in on/off-ramps can be written as:

$$\begin{aligned} \rho_{r_i}(k+1) &= \rho_{r_i}(k) + \frac{T}{\Delta_{r_i}}\Big(q^{\text{in}}_{r_i}(k) - r_i(k)\Big), \\ \rho_{s_i}(k+1) &= \rho_{s_i}(k) + \frac{T}{\Delta_{s_i}}\Big(s_i(k) - q^{\text{out}}_{s_i}(k)\Big), \end{aligned} \tag{20}$$

where $q^{\text{in}}_{r_i}$ and $q^{\text{out}}_{s_i}$ denote, respectively, the on-ramp inflow and off-ramp out-flow. Further, we have $r_i = h_{r_i}(\rho_{r_i})$, and $q^{\text{out}}_{s_i} = h_{r_i}(\rho_{s_i})$. In addition one can define $s_i = \beta_i q_i$, where β_i is the exit split ratio of cell i.

Now, to derive the overal dynamics of the system, let us define the aggregated vector of density of the cells as the system state. Thus, we have $x = (\rho_1, \ldots, \rho_n, \rho_{r_1}, \ldots, \rho_{r_n}, \rho_{s_1}, \ldots, \rho_{s_n})^T$, and the dynamics of the system can be written as:

$$\begin{aligned} \rho_1(k+1) &= \rho_1(k) + \frac{T}{\Delta_1\lambda_1}\Big(h_n(\rho_n(k)) - h_1(\rho_1(k)) + h_{r_1}(\rho_{r_1}(k)) - \beta_1 h_1(\rho_1(k))\Big), \\ \rho_2(k+1) &= \rho_2(k) + \frac{T}{\Delta_2\lambda_2}\Big(h_1(\rho_n(k)) - h_2(\rho_1(k)) + h_{r_2}(\rho_{r_2}(k)) - \beta_2 h_2(\rho_2(k))\Big), \\ &\vdots \\ \rho_n(k+1) &= \rho_n(k) + \frac{T}{\Delta_n\lambda_n}\Big(h_{n-1}(\rho_{n-1}(k)) - h_2(\rho_n(k)) + h_{r_n}(\rho_{r_n}(k)) - \beta_n h_n(\rho_n(k))\Big), \\ \rho_{r_1}(k+1) &= \rho_{r_1}(k) + \frac{T}{\Delta_{r_1}}\Big(q^{\text{in}}_{r_1}(k) - h_{r_1}(\rho_{r_1}(k))\Big), \\ &\vdots \\ \rho_{r_n}(k+1) &= \rho_{r_n}(k) + \frac{T}{\Delta_{r_n}}\Big(q^{\text{in}}_{r_n}(k) - h_{r_n}(\rho_{r_n}(k))\Big), \\ \rho_{s_1}(k+1) &= \rho_{s_1}(k) + \frac{T}{\Delta_{s_1}}\Big(\beta_1 h_1(\rho_1(k)) - h_{s_1}(\rho_{s_1}(k))\Big), \\ &\vdots \\ \rho_{s_n}(k+1) &= \rho_{s_n}(k) + \frac{T}{\Delta_{s_n}}\Big(\beta_n h_n(\rho_n(k)) - h_{s_n}(\rho_{s_n}(k))\Big), \end{aligned} \tag{21}$$

In order to investigate the controllabity of the system (21), that is a nonlinear dynamics, we aim to consider its linearized form around the steady state, and then use the methods for the controllability analysis of linear systems.

For $x \in \mathbb{R}^n$, assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function of x, and then consider the nonlinear system described as $\dot{x} = f(x)$. An equilibrium point x_{eq} is a state at which $f(x_{\mathrm{eq}}) = 0$. The linearized system is described as $\dot{x} = Ax$, where A is the Jacobian matrix of $f(\cdot)$, computed at x_{eq}.

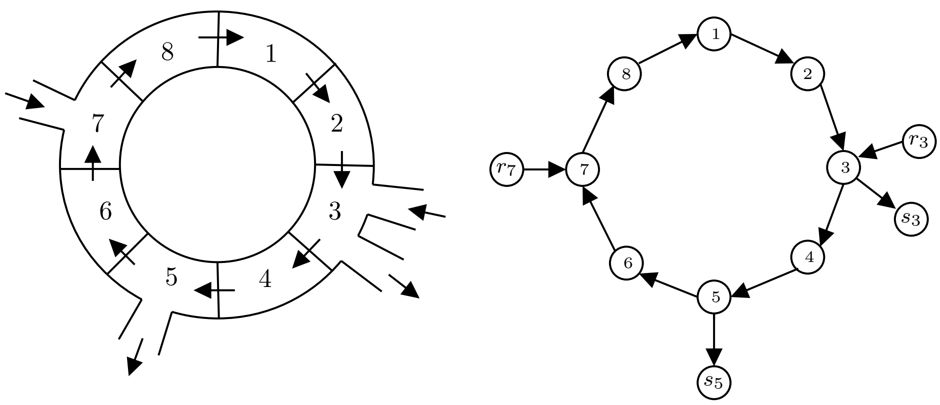
In this paper, we aim to study the structural controllablity of the system. In the structural controllability framework, the exact value of the enries of the system matrices as matrix A are not of interest; rather, only the zero-nonzoro pattern of these matrices is important. In fact, since $A_{ij} = \frac{\partial f_i}{\partial x_j}$, we have $A_{ij} \not\equiv 0$ if x_j appears in the x_i's differential equation.

We also not that the traffic density dynamics (21) is a self-damped dynamics, in the sense that for every $1 \leq i \leq n$, x_i appears in its own dynamics.

Now, let us define a graph $G = (V, E)$ associated with the system matrix A in $\dot{x} = Ax$. The sets $V = \{1, \ldots, n\}$ and $E \subset V \times V$ are, respectively, the node and the edge set of the graph. If $(i, j) \in E$, the node j (resp., i) is an out-neighbor (resp., in-neighbor) of i (resp., j). Any edge $(i, i) \in E$, $1 \leq i \leq n$, is a self-loop on node i. A simple graph is a graph that has no self-loop.

The graph G associated with the matrix A is a graph $G = (V, E)$, where $(i, j) \in E$ if and only if $A_{ji} \neq 0$. Since the network (21) is self-damped, we represent it by a simple graph. As an example, consider the traffic densities on a motorway ring road with the dynamics

Figure 2: (a) Model of a motorway ring road; (b) corresponding graph of the network.



(a) (b)

described in (21), that is shown in Fig. 2(a). The graph associated with this system is shown in Fig. 2(b).

Now, let us consider a linear time invariant (LTI) system with the following dynamics

$$x(k+1) = Ax(k) + Bu(k) \tag{22}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T$ is the vector of states of all nodes, and $u(t) = [u_1(t), \ldots, u_m(t)]^T$ is the vector of control inputs. The time-invariant matrix $A \in \mathbb{R}^{n\times n}$ is the state matrix, and G is a graph associated with A. Moreover, $B \in \mathbb{R}^{n\times m}$ is the time invariant input matrix, defined as $B = [e_{k_1}, \ldots, e_{k_m}]$. For $i = 1, \ldots, m$, the node k_i into which the input signal i is directly injected is called a *control node*. Similarly, we define the set of control nodes as $V_C = \{k_1, \ldots, k_m\}$. We define the set of control nodes as $V_C = \{k_1, \ldots, k_m\}$.

In the next parts, we determine a minimum set of control nodes V_C such that traffic density dynamics on a ring road network is weakly or strongly structurally controllable.

3.1 Weak structural controllability

Definition 1 *A network with dynamics (22) and a set of control nodes V_C is weakly structurally controllable if there exists at least one matrix A associated with the graph G for which the system is controllable.*

Based on this definition, the notion of weak structural controllability provides a necessary condition for the controllability of any network with the same structure. It has been shown that if a network is weakly structurally controllable, it would be controllable for *almost all* matrices A with the same zero-nonzero pattern (except for some pathological cases) Liu *et al.* (2011).

In order to provide an analysis for the weak structural controllability of the traffic network described in (21), some notions are defined in the following.

Definition 2 *A directed path is a sequence of nodes, e.g. $(i_1, i_2, \ldots, i_{p-1}, i_p)$, where for all $1 \leq j \leq p-1$, $(i_j, i_{j+1}) \in E$.*

Definition 3 *A matching in a graph is a set of edges that share no start or end nodes. A maximum matching is a matching of the largest cardinality. A node is referred to as a matched node if it is the end node of an edge in the maximum matching; otherwise, it is called an unmatched node.*

The weak structural controllability of a network can map into an equivalent geometrical problem as follows.

Theorem 1 (Liu *et al.* (2011)) *A network with graph G is weakly structurally controllable if and only if V_C includes any unmatched node in a maximum matching of G, and there exist directed paths from nodes of V_C to any node of G.*

Now, we aim to find the minimum number of control nodes for a ring road network, which can render it controllable. Then next result is an extension of the result of Mousavi and Kouvelas (2020) for the controllability analysis of highway networks.

Theorem 2 *Consider a highway network with dynamics (21), defined on a ring road. This network can be weakly structurally controllable by measuring only the density of one cell, and the single actuator can be placed at any cell that is not an off-ramp.*

Proof: Since this network is self-damped, there are self-loops on all of the nodes of the corresponding graph. Now, define $\mathcal{M}$ as the set of self-loops of all nodes. Since any two self-loops share no start or end nodes, from Definition 3, $\mathcal{M}$ is a maximum matching that has no unmatched node in the graph. Based on Theorem 1, it now suffices to find a set of nodes V_C from which there exist a directed path to every node of the graph. Considering the structure of a ring road (e.g., see graph in Fig. 2(b)), one can see that nodes corresponding to off-ramps have only one incoming edge. Thus, from these nodes, there exist no directed path to other nodes. However, by choosing any other node of the graph, one can reach to all other nodes via a directed path; this completes the proof.

3.2 Strong structural controllability

Definition 4 *A network with dynamics (22) and a set of control nodes V_C is strongly structurally controllable if for* all *matrices A associated with the same graph G the system* *is controllable.*

Thus, the notion of strong structural controllability ensures the controllability for all nonzero weights of connections; and by considering a fixed graph for the network, it provides a notion of robustness with respect to variations in the weights of the links.

Now, let us review some results regarding the strong structural controllability and its relation to the notion of zero forcing sets.

Consider a graph $G = (V, E)$ whose nodes are colored either black or white. The color of the nodes can be changed according to the *coloring rule*.

Definition 5 *If a black node $v \in V$ has only one white out-neighbor $u \in V$, it forces this node to become black; we designate this by $v \to u$. This rule is called a coloring rule.*

Next, the notion of a *zero forcing set* is defined.

Definition 6 *Let $Z \subset V$ be a set of initially black nodes of G. If after repeatedly applying the coloring rule, all nodes of G become black, Z is called a zero forcing set.*

There is a one-to-one correspondence between the control nodes that render a network strongly structurally controllable and zero forcing sets.

Theorem 3 (Monshizadeh *et al.* (2014); Reissig *et al.* (2014)) *A self-damped LTI network with graph G is strongly structurally controllable if and only if V_C is a zero forcing set of G.*

Now, in a ring road network, we aim here to obtain the smallest sets of control nodes that ensure its controllability for any set of nonzero parameters in (21). The next result is an extension of the results of Mousavi and Kouvelas (2020) for the strong structural controllability of highway networks defined on a ring-road.

Theorem 4 *Consider a ring road network described by the dynamics in (21). The minimum number of actuators to ensure the strong structural controllability of this network equals to* $\max\{1, n_r + n_s\}$*, where* n_r *and* n_s *are the number of on-ramps and off-ramps, respectively. In order to find the cells where these sensors should be placed, one should consider the following rules:*

a) If the ring road has no on/off-ramps, then the single actuator can be placed at any cell of the network.

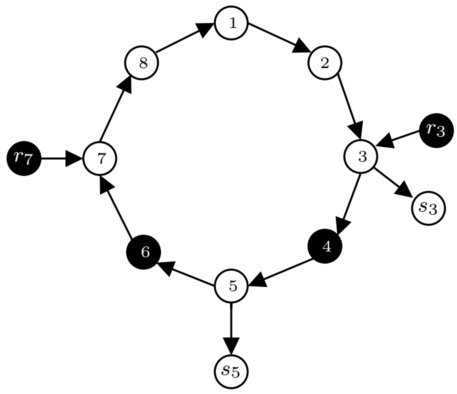
b) Associated with any off-ramp, one actuator should be placed at the cell immediately downstream of the cell that is connected to the off-ramp.

c) Regarding any on-ramp of the network, one sensor should be considered that can be placed at the on-ramp.

d) If there is a cell which is connected to both an off-ramp and on-ramp, one actuator is placed at the on-ramp, and one sensor is located at the cell immediately downstream of that.

Proof: Consider the simple graph associated with the network of the example depicted in Fig. 2(b). One can see that any off-ramp node s_i has only one in-neighbor, node i; while any on-ramp r_i has one one out-neighbor, node i. Moreover, any node i with an off-ramp and on-ramp, has out-neighbors, nodes $i+1$ and s_i, and in-neighbors, nodes $i-1$ and r_i, respectively. If an off-ramp node is initially black, based on the coloring rule in Definition 5, it cannot force any other node to become black. Thus, actuators should not be placed at off-ramp nodes. If the network has no on/off-ramp nodes, it is a directed cycle, where a zero forcing set can include any single node of the graph. If the network has one off-ramp s_i, one can see that $V_O = \{i+11\}$ is a zero forcing set. Now, assume that any other node of the graph initially becomes black (where the actuator is placed). Then, every node that has only one out-neighbor, can force it to become black. However, node i has two white out-neighbors, i.e. nodes s_i and $i+1$. Thus this node cannot change the color of its out-neighbors, and the coloring process is stopped. Thus, either s_i or $i+1$ should be initially black as well. In this case, the number of control nodes increases, and thus, the best place for the location of the sensor is node $i+1$. Now, consider a network with only one on-ramp r_i. One can see that $V_C = \{r_i\}$ is a zero forcing sets of the network, and according to Theorem 3, render it strongly structurally controllable. Since r_i has no in-neighbor, it cannot be forced by any other nodes of the graph to become black. Thus, if we initially color any other node of the graph, the node r_i cannot be black, and then the number of control nodes should increase; thus the proof of clause (c) is complete. Clause (d) can be proven in a similar way as well.

Figure 3: Configuration of the actuator placement in a ring road network.



For example, by considering the ring road network of Fig. 2(a) and employing Theorem 4, one can define $V_O = \{r_3, 4, 6, r_7\}$ as the smallest sets of control nodes, that ensures the system's strong structural controllability. This is illustrated in Fig. 3.

4 Control method

Thanks to the controllability analysis, we are able to determine the minimum number of actuators and their exact locations in highway networks. For employing these actuators in traffic flow management, it is crucial to develop a control method.
In this part, we utilized an MPC based control method designed by Chavoshi and Kouvelas (2020) for the modified METANET model.
In this work, the objective is to develop a coordinated control of VSL and RM in order to reduce congestion propagation and improve the traffic conditions on freeways. Non-Linear Model Predictive Control (NLMPC) is a straightforward control pathway to deal with nonlinear systems. It performs as a recursive on-line optimization of nonlinear problems subject to the system dynamics and additional constraints. In principle, the MPC type of controllers have the advantage of taking into account the impact of the predicted future behavior of the system into the current control signal designing. In this section, we briefly describe the structure of the MPC approach. MPC belongs to a larger group of control methods, known as Model Based Control. The common feature of this general type of control methods is that apart from considering the process model when designing the controller, the model is also considered in the block diagram. MPC has three fundamental features: it has a predictive model, it solves an optimization problem, and it performs as

a receding horizon control. The predictive model provides the relationship between the future inputs and future outputs. As the next step, a cost function is defined according to the finite future steps of input and output signals. MPC creates the control signals through optimizing the cost function. Finally, the first step of the control signal resulted from the optimization program is applied to the process and the same procedure is repeated for every time step (receding horizon). MPC has two main parameters, the prediction horizon k_p and the control horizon k_c, which denote the finite number of future outputs and inputs, respectively, that are considered in every iteration of optimization. Essentially, the control horizon is smaller or equal to the prediction horizon. In this study, we use METANET as the predictive model of macroscopic traffic flow on highway networks. In the modified version of METANET, equations (7)-(8), which describe the impacts of applying VSL and RM, contain minimum operators. In general, one can model the minimum operator by using an auxiliary integer variable in the optimization language. Therefore, in this problem the NLMPC is appeared as a recursive MINLP optimization. Either increasing the prediction horizon or the number of cells results in increasing the number of integer variables, and consequently the computational time grows exponentially. This drawback makes the designed NLMPC approach, impractical for large-scale real networks. In order to tackle this problem we propose to approximate the minimum operators with a continuous function. The work in Boyd and Vanderberghe (2004) has presented a log-sum-exp convex function $f(x) = \log(e^{x_1} + e^{x_2} + \cdots + e^{x_n})$ that is bounded as below:

$$\max(x_1, x_2, \ldots, x_n) \leq f(x) \leq \max(x_1, x_2, \ldots, x_n) + \log(n). \tag{23}$$

The above inequalities indicate that log-sum-exp function is an approximation of maximum operator, and thus the concept of a quasi-maximum function can be used. Quasi-maximum is a smooth approximation of the maximum operator:

$$\max(x_1, x_2, \ldots, x_n, \lambda) \simeq \frac{\log(e^{\lambda x_1} + e^{\lambda x_2} + \cdots + e^{\lambda x_n})}{\lambda}, \tag{24}$$

where λ is a setting parameter; larger λ increases the accuracy of the approximation. In this paper, we can prevent the usage of integer variables by applying the quasi-maximum to approximate the $\min\{\cdot\}$ operators in the predictive model. Therefore, the optimization transformes to a recursive NLP which requiers less computational time compared with the recursive MINLP version that is applied in Hegyi *et al.* (2005a). In order to justify this claim, we re-simulated the case study in Hegyi *et al.* (2005a) and applied the NLP method. The results indicate that the average computation time for one iteration of MPC is 1.13 s that is extremely faster than the 6 s reported by Hegyi *et al.* (2005a). Subsequently, we define a quadratic cost function form as follows:

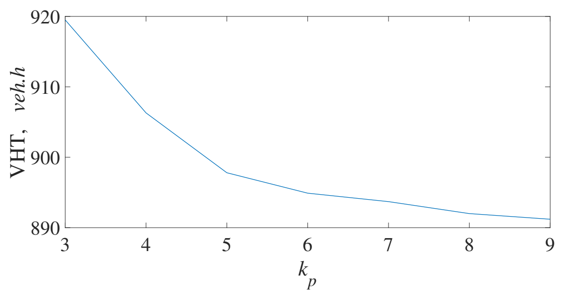
$$\min_{U(k),\ 1\leq k\leq k_c} \sum_{j=1}^{k_p-1}\sum_{i=1}^{N_\mathrm{s}}(\rho_i(j)-\rho_{\mathrm{cr,i}})^2+(v_i(j)-v_{\mathrm{cr,i}})^2+\omega\sum_{i=1}^{N_\mathrm{s}}(v_i(k_p)-v_{\mathrm{cr,i}})^2+(\rho_i(k_p)-\rho_{\mathrm{cr,i}})^2 \quad (25)$$

where, N_s is the number of segments included in the freeway stretch under study. In fact, the optimization problem can be described as searching for the control signals $U(k)$ (RM rate and speed limits) that minimize the error of the density and speed with their corresponded desired values. In equation (25) $\rho_{cr,i}$ and $v_{cr,i}$ are the desired densities and speeds, respectively, that correspond to the critical point in the FD of the segment i. The critical point represents the optimal functioning point of the traffic flow in each segment. Therefore, this cost function results in trying to regulate the traffic flow close to the critical point in the FD. We denote ω as a weight to emphasise the importance of the final state of the system at the end of the prediction horizon in the cost function. In this study, we consider the control horizon k_c equal to the prediction horizon k_p. By increasing the prediction horizon one can get more intuition about the future; therefore, the predicted trajectory is more accurate. However, increasing the prediction horizon will consequently increase the computational time and result in time-expensive solutions. In order to find an appropriate value we investigate the relationship between prediction horizon and performance of the controlled system. Vehicle Hours Traveled (VHT) is a well-known criterion to evaluate performance of traffic networks, and is computed as follows:

$$\mathrm{VHT} = T\sum_{j=1}^{K}\sum_{i=1}^{N_\mathrm{s}}\rho_i(j)v_i(j)\Delta(i,j) + T\sum_{j=1}^{K}\sum_{i=1}^{N_\mathrm{or}} w_i(j) \quad (26)$$

where K is the total simulation time. Figure 4 presents the effect of prediction horizon on VHT. By increasing the prediction horizon to more than 6 time steps, the improvement gained in terms of VHT is insignificant. In our experiments we choose to consider a prediction horizon of 9 time steps. Furthermore, in order to obtain reasonable solutions,

Figure 4: The effect of prediction horizon on VHT.



the states and control signals need to be bounded by physical and operational constraints. The constraints on the state variables are derived from the fundamental diagram and are as follows:

$$0 \leq \rho_i(k) \leq \rho_{i,\max} \tag{27}$$

$$0 \leq v_i(k) \leq v_{f,i} \tag{28}$$

$$0 \leq w_i(k) \tag{29}$$

We also define constraints on the control signals. Since the implementation of RM cannot block the on-ramp inflow completely, the RM rate is restricted by a lower bound ($c_{\min} = 0.2$), which is the minimum admissible value. On the other hand, the speed limit has also lower and upper bounds. Obviously, the speed limit cannot exceed the free flow speed; in addition, a lower bound ($v_{\min} = 50$ km/h) is considered due to safety reasons.

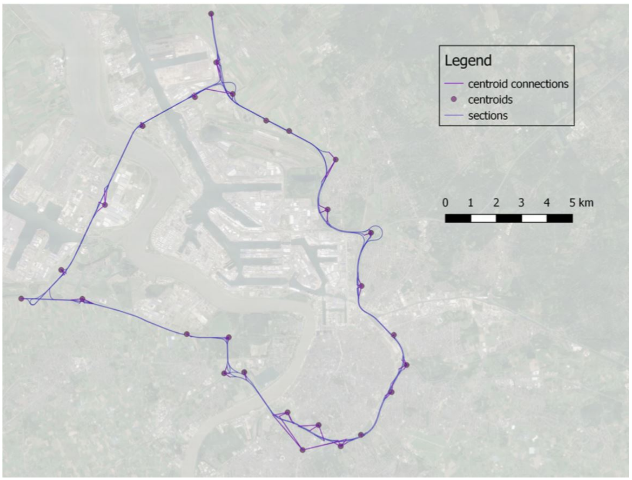
$$C_{\min} \leq c_i(k) \leq 1 \tag{30}$$

$$v_{\min} \leq v_{VSL,i} \leq v_{f,i} \tag{31}$$

5 Simulation setup

The controllability analysis enables us to apply the designed NLMPC control method for large networks on real-time. However, considering the fact that we investigate controllability for a simplified linear version of the macroscopic traffic model, we expect some inexactness in the analysis's results. Therefore, by employing the results of controllability we would observe a trade-off between the control's performance and the computational costs. As the ultimate goal, we would like to assess this trade-off by performing a simulation experiment for a large network. We have selected the Antwerp ring road as a

Figure 5: An overview to the Antwerp ring road Mattas *et al.* (2018)



case study for the simulation. Antwerp is a city in Belgium, has the second biggest port in Europe. Due to this fact, Antwerp ring road serves high traffic demand that is a mixture of heavy vehicles traffic due to the port and the normal commuter's demand for the city. Antwerp experienced heavy congestion during rush hours. You can observe the Antwerp topology in the following figure. We focus on the counter clockwise direction to perform the simulation. Antwerp has roughly the 48 km length with 27 centroids (destination/origin points) distributed along this length. Mattas *et al.* (2018) designed the OD matrix based on the real data captured from the Antwerp's detectors and simulated the traffic flow for morning peak hours in aimsun. We would use this OD matrix to design the demand for the network and simulate the macroscopic model of Antwerp traffic during the morning rush hours. In order to apply the cell transmission based macroscopic model, we did the segmentation by taking into account the CFL condition, Resulted in 139 cells along the Antwerp.

6 Future work

As the next step, it is essential to calibrate the METANET's parameters with the help of aimsun simulation for Antwerp network. After finishing the initial prepration of Antwerp ring road, we are ready to evaluate the performance of the proposed strategy. For this purpose, we designed two simulation scenarios for Antwerp case study. In the first scenario, we ignore the controllability analysis and employ VSL for every cell. Based on our experience in smaller networks, although applying VSL for every cell enhances traffic flow features significantly, it is not applicable in real-time because of the computational time issue. In the second scenario, we place the VSLs in the Antwerp network in accordance with the results of controllability analysis. Decreasing the number of VSLs notably simplify the optimization problem. In addition, since we locate VSLs based on the controllability analysis, we envision a real-time control management that maintains roughly the performance of the basic scenario.

7 References

Agarwal, S., P. Kachroo and S. Contreras (2015) A dynamic network modeling-based approach for traffic observability problem, *IEEE Trans. Intell. Transp. Syst.*, **17** (4) 1168–1178.

Bekiaris-Liberis, N., C. Roncoli and M. Papageorgiou (2017a) Highway traffic state estimation per lane in the presence of connected vehicles, *Transp. Res. B: Methodol.*, **106**, 1–28.

Bekiaris-Liberis, N., C. Roncoli and M. Papageorgiou (2017b) Structural observability of multi-lane traffic with connected vehicles, paper presented at the *IEEE ITSC*, 1–8.

Boyd, S. and L. Vanderberghe (2004) *Convex Optimization*, 1st edn., Combridge university Press, The Edinburgh Building, Cambridge, CB2 8RU, UK.

Carlson, R. C., I. Papamichail, M. Papageorgiou and A. Messmer (2010) Optimal mainstream traffic control of large-scale motorway networks, *Transportation Research Part C: Emerging Technologies*, **18**, 193–212.

Castillo, E., A. Calviño, H. K. Lo, J. M. Menéndez and Z. Grande (2014) Non-planar hole-generated networks and link flow observability based on link counters, *Transp. Res. B: Methodol.*, **68**, 239–261.

Castillo, E., I. Gallego, S. Sanchez-Cambronero and A. Rivas (2010) Matrix tools for general observability analysis in traffic networks, *IEEE Trans. Intell. Transp. Syst.*, **11** (4) 799–813.

Castillo, E., P. Jimenez, J. M. Menendez and A. J. Conejo (2008) The observability problem in traffic models: algebraic and topological methods, *IEEE Trans. Intell. Transp. Syst.*, **9** (2) 275–287.

Chavoshi, K. and A. Kouvelas (2019) A feedback linearization approach for closed-loop control design: Application to variable speed limits and ramp metering, paper presented at the *8th Symposium of the European Association for Research in Transportation (hEART 2019)*.

Chavoshi, K. and A. Kouvelas (2020) Nonlinear model predictive control for coordinated traffic flow management in highway systems, paper presented at the *2020 European Control Conference*, Saint Petersburg, Russia, May 2020.

Contreras, S., P. Kachroo and S. Agarwal (2015) Observability and sensor placement problem on highway segments: A traffic dynamics-based approach, *IEEE Trans. Intell. Transp. Syst.*, **17** (3) 848–858.

Haj-Salem, H., N. Farhi, J. P. Lebacque and N. Bhouri (2018) Development of coordinated ramp-metering based on multi-objective nonlinear optimization functions: Traffic and safety, paper presented at the *Advanced Concepts, Methodologies and Technologies for Transportation and Logistics*, 334–354, Cham.

Hegyi, A., B. D. Schutter and H. Hellendoorn (2005a) Model predictive control for optimal coordination of ramp metering and variable speed limits, *Transportation Research Part C: Emerging Technologies*, **13**, 185–209.

Hegyi, A., B. D. Schutter and J. Hellendoorn (2005b) Optimal coordination of variable speed limits to suppress shock waves, *IEEE Transactions on Intelligent Transportation Systems*, **6** (1) 102 – 112.

Khondaker, B. and L. Kattan (2015) Variable speed limit: A microscopic analysis in a connected vehicle environment, *Transportation Research Part C: Emerging Technologies*, **58**, 146–159.

Kotsialos, A., M. Papageorgiou and F. Middelham (2001) Optimal coordinated ramp metering with advanced motorway optimal control, *Transportation Research Record Journal of the Transportation Research Board*, **1748** (1) 55–65.

Li, D., P. Ranjitkar and A. Ceder (2014) Integrated approach combining ramp metering and variable speed limits to improve motorway performance, *Transportation Research Record Journal of the Transportation Research Board*, **2470**, 86–94.

Liu, Y.-Y., J.-J. Slotine and A.-L. Barabási (2011) Controllability of complex networks, *Nature*, **473** (7346) 167–173.

Lu, C., J. Huang, L. Deng and J. Gong (2017) Coordinated ramp metering with equity consideration using reinforcement learning, *Journal of Transportation Engineering, Part A: Systems*, **143**.

Mattas, K., M. Makridis, P. Hallac, M. Alonso Raposo, C. Thiel, T. Toledo and C. Biagio (2018) Simulating deployment of connectivity and automation on the antwerp ring road, *IET Intelligent Transport Systems*, **12** (9) 1036–1044.

Monache, M. L. D., B. Piccoli and F. Rossi (2017) Traffic regulation via controlled speed limit, *SIAM Journal on Control and Optimization, Society for Industrial and Applied Mathematics*, **55** (5) .2936–2958.

Monshizadeh, N., S. Zhang and M. K. Camlibel (2014) Zero forcing sets and controllability of dynamical systems defined on graphs, *IEEE Trans. Automat. Contr.*, **59** (9) 2562–2567.

Mousavi, S. S., M. Haeri and M. Mesbahi (2018) On the structural and strong structural controllability of undirected networks, *IEEE Trans. Automat. Contr.*, **63** (7) 2234–2241.

Mousavi, S. S., M. Haeri and M. Mesbahi (2021a) Laplacian dynamics on cographs: Controllability analysis through joins and unions, *IEEE Trans. Automat. Contr.*, **66** (3).

Mousavi, S. S., M. Haeri and M. Mesbahi (2021b) Strong structural controllability of networks under time-invariant and time-varying topological perturbations, *IEEE Trans. Automat. Contr.*, **66** (3).

Mousavi, S. S. and A. Kouvelas (2020) Structural observability of traffic density dynamics on a motorway ring road, paper presented at the *the 23rd IEEE International Conference on Intelligent Transportation Systems*, Rhodes, Greece, September 2020.

Papageorgiou, M., H. S. Habib and J. M. Blosseville (1991) Alinea: A local feedback control law for on-ramp metering, *Transportation Research Record Journal of the Transportation Research Board*, **1320**, 58–64.

Papamichail, I., A. Kotsialos, I. Margonis and M. Papageorgiou (2010) Coordinated ramp metering for freeway networks -a model predictive hierarchical control approach, *Transportation Research Part C: Emerging Technologies*, **18** (3) 311–331.

Reissig, G., C. Hartung and F. Svaricek (2014) Strong structural controllability and observability of linear time-varying systems, *IEEE Trans. Automat. Contr.*, **59** (11) 3087–3092.

Rinaldi, M. (2018) Controllability of transportation networks, *Transport. Res. Part B: Methodological*, **118**, 381–406.

Zhang, Y. and P. A. Ioannou (2017) Coordinated variable speed limit, ramp metering and lane change control of highway traffic, *IFAC Papers OnLine*, **50** (1) 5307–5312.